

# A class of groups with inert subgroups \*

*to the memory of Jim Wiegold*

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**Abstract.** Two subgroups  $H$  and  $K$  of a group are commensurable iff  $H \cap K$  has finite index in both  $H$  and  $K$ . We prove that hyper-(abelian or finite) groups with finite abelian total rank in which every subgroup is commensurable to a normal one are finite-by-abelian-by-finite.

Keywords: *normal, commensurable, inert, finite index, subgroup.*

In [2] authors study CF-groups (core finite), i.e. groups  $G$  in which  $|H : H_G|$  is finite for each subgroup  $H$ . In other words, each  $H$  contains a normal subgroup of  $G$  with finite index in  $H$ . This class arises in a natural way as the dual of the class of groups  $G$  with  $|H^G : H|$  finite for each  $H \leq G$ . The latter class was earlier considered in a very celebrated paper of B.H. Neumann [8] and revealed to be the class of finite-by-abelian groups, i.e. groups with finite derived group. In fact in [2] it is proved that a locally finite CF-groups are abelian-by-finite (i.e. they have an abelian subgroup with finite index) and are BCF too. This means that they are CF in a bounded way, i.e. the above index is bounded independently of  $H$ . As Tarski groups are CF, a complete classification of CF-groups seems to be much difficult. Anyway, in [12] it is proved that a locally radical CF-group is abelian-by-finite indeed, while an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given.

To consider the above two classes in the same framework, we consider *CN-groups*, that is groups in which each subgroup  $H$  is *cn* (commensurable

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\*these results have been communicated at ISCHIA GROUP THEORY 2012 Conference, Ischia (Naples, Italy) March, 26th - 29th. Also Preprint 10/2012 at Dipartimento Matematica e Appl. "R. Caccioppoli", Napoli

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to a normal one). Recall that two subgroups  $H$  and  $K$  of a group  $G$  are told *commensurable* iff  $H \cap K$  has finite index in both  $H$  and  $K$ . This is an equivalence relation and will be denoted by  $\sim$ . Clearly, if  $H \text{ cn } G$ , then  $H$  is *inert* in  $G$ , that is commensurable with all conjugates of its. Groups whose all subgroups are inert are called *inertial* (or also *totally inertial*) groups and have received much attention (see [1] and [11]).

From above quoted results following questions arise.

**Question 1:** Given a group theoretical class  $\mathbf{X}$ , is a CN-group in  $\mathbf{X}$  finite-by-abelian-by-finite?

We show that *Question 1 has a positive answer for the class of hyper-(abelian or finite) groups with finite abelian total rank*. As costumary we denote this class by  $\mathfrak{S}_1\mathfrak{F}$ .

**Theorem 1** *A CN-group  $G$  in the class  $\mathfrak{S}_1\mathfrak{F}$  is finite-by-abelian-by-finite.*

**Question 2:** When is a finite-by-abelian-by-finite CN-group BCN?

Recall that BCN means CN in a bounded way, that is

$\exists n \in \mathbb{N} \forall H \leq G \exists N \triangleleft G$  such that  $|H/(H \cap N)| \cdot |N/(H \cap N)| \leq n$  (and  $n$  is independent of  $H$ ). This is true for CN-groups  $G$  that following [11] we call of *elementary type* that is with normal subgroups  $G_1 \leq G_0$  of  $G$ , finite and of finite index in  $G$  resp., such that elements of  $G$  act as power automorphisms on the abelian factor  $G_0/G_1$ . Note that such a group is clearly BCN, as each subgroup  $H$  of  $G$  is commensurable to  $N := (H \cap G_0)G_1$  and  $|H/(H \cap N)| \cdot |N/(H \cap N)| \leq |G/G_0| \cdot |G_1|$ . Recall that an automorphism is said *power* iff it fixes setwise each subgroup.

In Theorem 2 we see that *for abelian-by-finite groups the two classes CN and CF do coincide*, and their structure follows from results in [6]. Theorem 2 also gives complete description of abelian-by-finite BCF-groups, which are precisely abelian-by-finite BCN-groups.

For terminology, notation and basic facts we refer to [10] and [11]. Recall that soluble-by-finite groups with finite abelian total rank are precisely *groups  $G$  with a series whose factors are either subgroups of direct products of finitely many either Prüfer groups or copies of  $\mathbb{Q}$ , the last factor being possibly a finite non-solvable group* (see [7]). Notice that abelian  $p$ -sections of such a group are Chernikov. Also if  $D = \text{Div}(G)$  is the biggest normal abelian divisible periodic subgroup of  $G$ , then  $G/D$  has a finite series whose factors are finite

or copies of  $\mathbb{Q}$ . If  $G$  is CN, and hence totally inertial, then its elements act on  $D$  as power automorphisms (see [11]).

**Proof of Theorem 1** Since the set  $D = \text{Div}(G)$  is a Chernikov group, we may factor out by  $D_{p'}$  ( $p \in \pi(D)$ ) and assume that  $D$  is a  $p$ -group of finite rank. In fact suppose that for each prime  $p$  in the finite  $\pi := \pi(D)$ ,  $G/D_{p'}$  is finite-by-abelian-by-finite. Then  $G$  has a subgroup of finite index  $G_p \geq D_{p'}$  such that  $G'_p D_{p'}/D_{p'}$  is finite. So  $H := \bigcap_{p \in \pi} G_p$  has finite index in  $G$  and  $H'/H' \cap D_{p'}$  is finite for any  $p \in \pi$ . Thus  $H'$  is finite and  $G$  itself is finite-by-abelian-by-finite.

So assume that  $D$  is a  $p$ -group of finite rank. By results of Robinson [11], if  $G$  is not elementary, there are a finite normal subgroup  $F$  of  $G$  and a normal subgroup  $K$  of finite index of  $G$  such that  $K/DF$  is finite-by-(torsion-free abelian) and either  $K/F$  splits on  $\text{Div}(K/F)$  (type I in [11]), or  $\text{Div}(K/F) \leq Z(K/F)$  (type II in [11]).

We may assume  $F = 1$  and  $K = G$ , and so  $G/D$  is finite-by-(torsion-free abelian) and either  $G$  splits on  $D$  or  $D \leq Z(G)$ .

In the former case  $G = D \rtimes Q$ , and there is a finite normal subgroup  $L$  of  $Q$  such that  $Q/L$  is torsion-free abelian. Let  $g \in Q$ ,  $H := \langle g \rangle$  and let  $N \triangleleft G$  commensurable to  $H$ . Then  $[H \cap N, D] \leq N$ . If  $[H \cap N, D] \neq 1$ , then  $[H \cap N, D] = D$  (recall that elements of  $G$  act as power automorphisms on  $D$ ) and hence  $D \leq N$ , contradicting the fact that  $H \sim N$ . Hence  $[H \cap N, D] = 1$ , and so there is a subgroup of finite index of  $H$  centralizing  $A$ . Hence  $Q/C_Q(D)$  is finite, being a periodic group of automorphisms of the Chernikov group  $D$ . We may assume now that  $Q = C_Q(D)$  and so  $G = D \times Q$  is finite-by-abelian, as wished.

In the latter case ( $D$  central in  $G$ ), we claim that  $G'$  is finite. We will show that  $|H^G : H|$  is finite for any  $H \leq G$ , so by the above quoted result of B.H. Neumann (see [8]),  $G'$  is finite. We may assume  $D \not\leq H$ , and  $D \cap H = 1$ , as it is normal in  $G$  and we can factor out by it. So  $H'$  is finite. Take  $N \triangleleft G$  commensurable to  $H$  and let  $H_1 := H \cap N$ . As  $|H : H_1|$  is finite,  $H^n \leq H_1 H'$  for a suitable  $n \in \mathbb{N}$ . As  $G$  is nilpotent of class 2,  $[H_1 H', G] = [H_1, G] \leq N \cap D$ , which is finite (as  $H \cap D = 1$  and  $H \sim N$ ). Moreover  $[H, G]^n = [H^n, G] \leq [H_1 H', G]$  is finite, too. As  $[H, G] \leq D$  has finite rank, we have that  $[H, G]$  is finite and so  $|H^G : H|$  is finite, as wished.  $\square$

Recall that an automorphism  $\gamma$  of a group  $G$  setwise mapping each subgroup to a commensurable one is told *inertial*. Moreover  $\gamma$  is called a *bound-*

*edly inertial*, or simply *bin* iff  $\exists n \in \mathbb{N} \quad |H/(H \cap H^\gamma)| \cdot |H^\gamma/(H \cap H^\gamma)| \leq n$  for all subgroup  $H$  of  $G$  (and  $n$  is independent of  $H$ ). Such automorphisms are studied in [5] in the case  $G$  is abelian. Clearly, a group  $G$  is a finite-by-abelian CN (BCN, resp.) iff  $G$  acts by means of a finite group of inertial (bin, resp.) automorphisms on an abelian normal subgroup  $A$  of its. Notice that periodic inertial automorphisms  $\gamma$  are *almost-power*, that is  $\forall H \leq A$  there is a  $\gamma$ -invariant subgroup of  $H$  with finite index.

**Theorem 2** *Let  $G$  be an abelian-by-finite group.*

- *$G$  is CN (resp. BCN) iff it is CF (resp. BCF).*
- *If  $G$  is periodic, then  $G$  is a BCF-group iff  $G$  is a CF-group .*
- *If  $G$  is non-periodic, then  $G$  is a BCF-group iff there is a normal series*

$$1 \leq V \leq K \leq A \leq G$$

*where:*

- i)  $A$  is abelian with finite index,*
- ii)  $G/K$  has finite exponent,*
- iii) each element of  $G$  acts on  $K$  as the identity or the inversion map,*
- iv)  $V$  is free abelian ,*
- v) each element of  $G$  acts on periodic group  $A/V$  as an almost-power automorphism,*
- vi) either  $K = A$  (elementary case) or  $V$  has finite rank.*

**Proof.** Let  $A$  be an abelian normal subgroup of  $G$ . The group  $G$  acts as a finite group of automorphisms of  $A$  and hence the properties CN and CF (BCN and BCF, resp.) are obviously equivalent, and they are equivalent to the fact that every element  $g$  of  $G$  acts on  $A$  as an inertial automorphism.

Let now  $G$  be BFC. If  $G$  is periodic, by results in [5] and [6] it follows that there is  $n$  such that  $|H/H_G| \leq n$  for each  $H \leq G$ .

If  $G$  is a non-periodic CF-group, then  $G$  is BCF iff every element  $g$  of the finite group  $\bar{G} = G/C_G(A)$  acts as a bin-automorphism on  $A$ . By Th. 3 and Cor.1 of [5], this is equivalent to saying that for each  $g$  the subgroup  $E_g = A^{g-\epsilon}$  has finite exponent (where  $g = \epsilon = \pm 1$  on  $A/T$ , where  $T \neq A$  is the torsion subgroup of  $A$ ). Take  $E := \langle E_g \mid g \in \bar{G} \rangle$  and, for any  $g \in \bar{G}$ , let  $K_g$  be the kernel of the endomorphism  $g - \epsilon$  of  $A$  and put  $K := \bigcap_{g \in \bar{G}} K_g$ . As  $E_g$  is the image of  $g - \epsilon$ , it is clear that  $E$  has finite exponent iff  $A/K$  has. Moreover  $K$  is  $G$ -hamiltonian. The statement follows from [5], Th. 3 and [6], as  $V \leq K$ .  $\square$

## References

- [1] V.V. Belayev, M. Kuzucuoğlu and E. Seckin, Totally inert groups, *Rend. Sem. Mat. Univ. Padova* **102** (1999), 151-156.
- [2] J.T. Buckley, J.C. Lennox, B.H. Neumann, H. Smith and J. Wiegold, Groups with all subgroups normal-by-finite. *J. Austral. Math. Soc. Ser. A* **59** (1995), no. 3, 384-398.
- [3] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, *Rend. Sem. Mat. Univ. Padova* **81** (1989), 107-149.
- [4] C. Casolo, Subgroups of Finite Index in Generalized T-groups, *Rend. Sem. Mat. Univ. Padova* **80** (1988), 265-277.
- [5] U. Dardano and S. Rinauro, Inertial automorphisms of an abelian group, to appear on *Rend. Sem. Mat. Univ. Padova*.
- [6] S. Franciosi, F. de Giovanni and M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.* **23(14)** (1995), 5483-5497.
- [7] J.C. Lennox and D.J.S. Robinson, "The theory of infinite Soluble groups", Oxford, 2004.
- [8] B.H. Neumann, Groups with finite classes of conjugate subgroups *Math. Z.* **63** (1955), 76-96. Springer V., Berlin, 1972.
- [9] D.J.S. Robinson, Applications of cohomology to the theory of groups [in Groups - St. Andrews 1981], Cambridge Univ. Press, Cambridge-New York, 1982.
- [10] D.J.S. Robinson, "A Course in the Theory of Groups", Springer V., Berlin, 1982.
- [11] D.J.S. Robinson, On inert subgroups of a group, *Rend. Sem. Mat. Univ. Padova* **115** (2006), 137-159.
- [12] H. Smith and J. Wiegold, Locally graded groups with all subgroups normal-by-finite, *J. Austral. Math. Soc. Ser. A* **60** (1996), no. 2, 222-227.

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